What we need from Topic 59

Problems on ages give the ratio, sum, difference or average of ages at one or more time points and ask for an age or a number of years. They rest on one fact: every person grows older by the same number of years, so the difference between two ages never changes, while the ratio of two unequal ages does.

A ratio does not give the actual quantities.

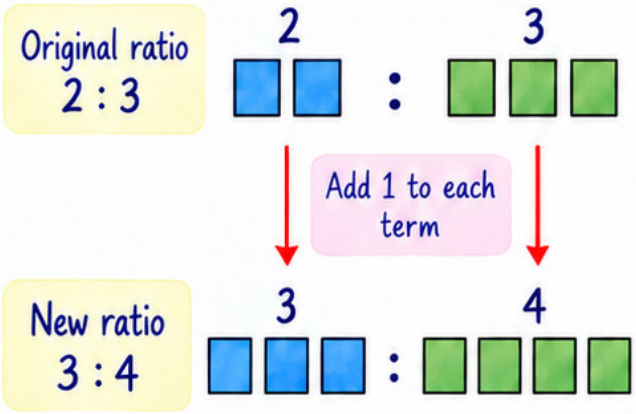
If two ages are in the ratio $a : b$, they are ak and bk for some positive number k.

Person A	Person B
Age = ak	Age = bk

The ratio of their ages is $a : b$, which means the ages are ak and bk for some positive number k.

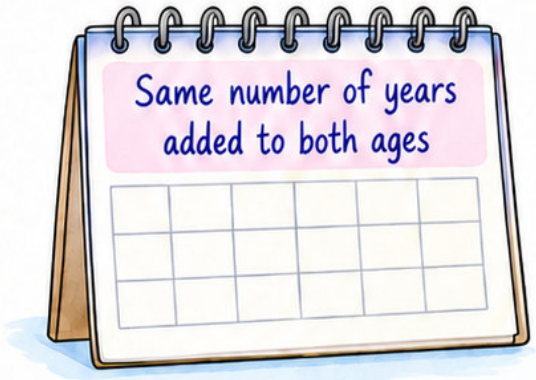
Adding the same number changes the ratio.

When the two terms are unequal, adding the same non-zero number to both changes the ratio: $2 : 3$ is not equal to $3 : 4$. This is why the ratio of two ages changes with time.



The difference between two ages never changes.

If two people are ak and bk years old, the difference between their ages is $|bk - ak| = |b - a|k$. After t years, both are t years older, so the difference is still $|b - a|k$.



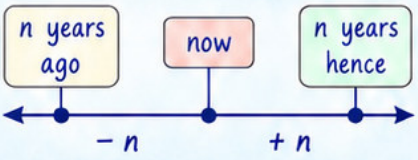
Person A	Person B
Age = ak	Age = bk
$+ n$	$+ n$
Age = $ak + n$	Age = $bk + n$

Sources: Day 16 Source Index | Topic 158 | Section 2.1

The basic facts

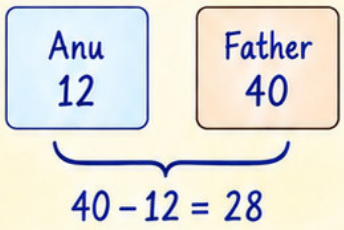
Fix the reference point

Fix "now" as the reference point. "n years ago" means present age − n; "n years hence" means present age + n. Do this for every person.



The difference of two ages never changes.

Anu is 12 and her father is 40 now. The difference is 28. In 8 years they will be 20 and 48; the difference is still 28. Her father was 28 when Anu was born: at the birth of the younger person, the elder's age equals the difference.



The ratio of two unequal ages changes.

Father to Anu: 8 years ago 32 : 4 = 8 : 1; now 40 : 12 = 10 : 3; in 8 years 48 : 20 = 12 : 5. The ratio of the elder's age to the younger's keeps falling towards 1 (8, then 3.33..., then 2.4), but it never reaches 1.

8 : 1 (= 8) → 10 : 3 (= 3.33....) → 12 : 5 (= 2.4)

8 years ago	Now	8 years hence
32 : 4	40 : 12	48 : 20

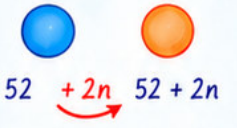
The table of ages, differences, sums and ratios

Time point	Elder	Younger	Difference	Sum	Ratio
8 years ago	32	4	28	36	8 : 1
Now	40	12	28	52	10 : 3
8 years hence	48	20	28	68	12 : 5

A sum grows by n for each person.

In n years the sum of two ages grows by 2n, and the total age of a group of p persons grows by p × n. The average age of a group whose members do not change grows by n.

Two people: sum increases by 2n.



Group of p persons: total increases by p × n.

p members

Total T → +p×n → T + p×n

Average age:
The average age of a group whose members do not change grows by n.

A birthday example



Ages are usually whole years.

Ages in these problems usually come out in whole years: re-check any fractional answer; a negative age always shows a slip.

Summary of key ideas

- Fix "now" as the reference point. "n years ago" means present age − n; "n years hence" means present age + n. Do this for every person.
- The difference of two ages never changes.
- The ratio of two unequal ages changes.
- A sum grows by n for each person.
- Ages in these problems usually come out in whole years: re-check any fractional answer; a negative age always shows a slip.

Sources: Day 16 Source Index | Topic 158 | Section 2.2

The ratio-parts method

General method

Write the present ages as ak and bk. After n years they are $(ak + n)$ and $(bk + n)$; n years ago they were $(ak - n)$ and $(bk - n)$. Set the new pair equal to the given ratio $c : d$, cross-multiply and solve for k. This is a linear equation in one unknown.

n years ago

Ages were $(ak - n)$ and $(bk - n)$.

Present ages

Ages now are ak and bk.

After n years

Ages will be $(ak + n)$ and $(bk + n)$.

Example.

The present ages of P and Q are in the ratio 7 : 5.
After 8 years the ratio will be 9 : 7.
Ages now: $7k$ and $5k$.
Then $(7k + 8) : (5k + 8) = 9 : 7$, so $7(7k + 8) = 9(5k + 8)$,
that is, $49k + 56 = 45k + 72$, so $4k = 16$ and $k = 4$.
P is 28 and Q is 20.
Check: after 8 years, 36 : 28 = 9 : 7.



A bar model using ratio parts

										Total age
P now (7 parts)	4	4	4	4	4	4	4			28 years
Q now (5 parts)	4	4	4	4	4					20 years
P after 8 years	4	4	4	4	4	4	4	4	4	36 years
Q after 8 years	4	4	4	4	4	4	4			28 years

Key

4 — Each filled cell represents 4 years.

4 — The two yellow cells show the added 8 years ($n = 8$).

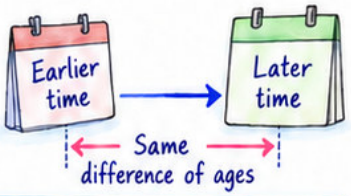
Equation steps (as in the example)

$(7k + 8) : (5k + 8) = 9 : 7$ → $7(7k + 8) = 9(5k + 8)$ → $49k + 56 = 45k + 72$ → $4k = 16$, $k = 4$ → P is 28, Q is 20. Check: 36 : 28 = 9 : 7.

Sources: Day 16 Source Index | Topic 158 | Section 2.3

The difference-of-ages method

The difference of the ages is the same at both time points, so it can be measured in parts. If the two ratios have the same difference of parts, one part means the same number of years in both, and the change in one person's parts is the number of years that passed.



Same difference.

In 2.3, 7 : 5 and 9 : 7 both have a difference of 2 parts. P goes from 7 parts to 9 parts, that is, 2 parts = 8 years, so 1 part = 4 years. P = 7 × 4 = 28 and Q = 5 × 4 = 20.

At the earlier time (ratio 7 : 5)

Person	Earlier age	Later age	Added years
P	28	36	8
Q	20	28	8

Ratios 7 : 5 → 9 : 7; difference 2 parts at both times; 2 parts = 8 years.

2 parts = 8 years so 1 part = 4 years

At the later time (ratio 9 : 7)

Person	Earlier age	Later age	Added years
P	28	36	8
Q	20	28	8

P increases by 2 parts (from 7 to 9), so 2 parts = 8 years and 1 part = 4 years.

Different differences: make them equal first.

The present ages of R and S are in the ratio 3 : 2, and 8 years ago they were in the ratio 7 : 4. The differences of parts are 1 and 3. Multiply 3 : 2 by 3 to get 9 : 6 (difference 3). Now R goes from 7 parts (8 years ago) to 9 parts (now): 2 parts = 8 years, so 1 part = 4 years. R = 9 × 4 = 36 and S = 6 × 4 = 24. Check: 8 years ago 28 : 16 = 7 : 4.

Present ratio 3 : 2

3 : 2 (difference 1)

Multiply 3 : 2 by 3 to get 9 : 6 (difference 3).

Equivalent present ratio 9 : 6

9 : 6 (difference 3)

This is an equivalent ratio with the same ages, not a change in ages.

Same ages, different choice of ratio units.

8 years ago (ratio 7 : 4)

Person	Age	Ratio
R	28	7
S	16	4

R = 9 × 4 = 36
S = 6 × 4 = 24

+ 8 years to each

Now (ratio 9 : 6)

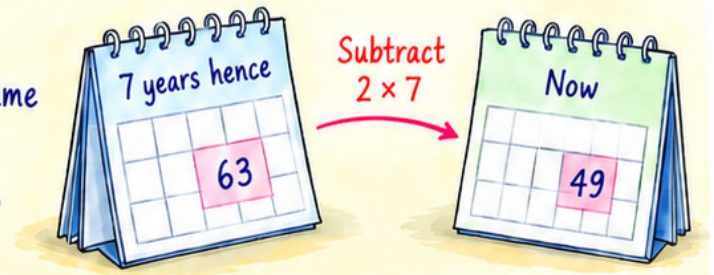
Person	Age	Ratio
R	36	9
S	24	6

Part differences: 3 at both time points; 2 parts = 8 years; 1 part = 4 years.
Check: 8 years ago 28 : 16 = 7 : 4.

The sum-of-ages and difference-of-ages data

Sum given.

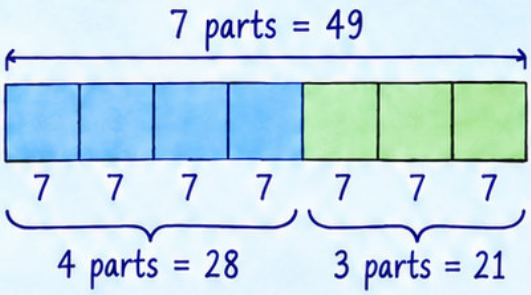
If the ratio is $a : b$ and the sum at the same time point is S, then $k = S \div (a + b)$.
If the sum is given at another time point, first move it to the time of the ratio (2n for two persons).



Future sum = 63 Present sum = 49

Example:

The ages of two sisters are in the ratio 4 : 3 now, and 7 years hence the sum of their ages will be 63.
The present sum is $63 - 2 \times 7 = 49$.
Then 7 parts = 49, so $k = 7$:
the ages are 28 and 21.
Check: $35 + 28 = 63$.

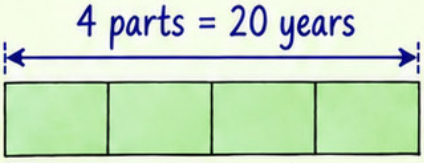


Ages are 28 and 21.

Difference given.

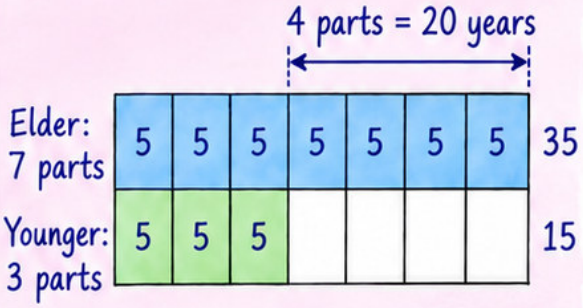
If the ratio is $a : b$ and the elder is D years older, then $(a - b)$ parts = D years.

$7 - 3 = 4$ parts
4 parts = 20 years
1 part = 5 years



Example:

Ratio 7 : 3 and a difference of 20 years.
4 parts = 20, so $k = 5$:
the ages are 35 and 15.



Sources: Day 16 Source Index | Topic 158 | Section 2.5

The average-of-ages method

Total age = average x number of members.

Total age = average x number of members.



Average age = 17
Number of members = 5

Total age = 17 x 5 = 85

After n years, add (number of members x n) to the old total.

After n years, add (number of members x n) to the old total.



New total = old total + (number of members x n)

A member joins.

Three years ago the average age of a family of 5 was 17 years. A baby has been born since then, and today the average age of the 6 members is 17 years.

Total 3 years ago = 5 x 17 = 85.
The same 5 persons today = 85 + 5 x 3 = 100.
Total of the 6 today = 6 x 17 = 102.
The baby is 102 - 100 = 2 years old.

The average is the same as before, yet the baby is not 17: the old members have grown older.

Three years ago:
5 members, average age = 17



Total = 5 x 17 = 85

3 years

Today:
6 members, average age = 17



Total = 6 x 17 = 102

The same 5 persons today:



+



Baby:
102 - 100 = 2 years old.

Total = 85 + 5 x 3 = 100

A member leaves.

The average age of 7 members of a group is 26 years. One member leaves today, and the average of the other 6 is 24 years.

The member who left is
7 x 26 - 6 x 24 = 182 - 144
= 38 years old.

7 members,
average age = 26



Total = 7 x 26 = 182

One member leaves

6 members,
average age = 24



Total = 6 x 24 = 144

The member who left is 7 x 26 - 6 x 24 = 182 - 144 = 38 years old.

Day 62 (Topic 618) takes up averages and sums of ages in more detail.

Sources: Day 16 Source Index | Topic 158 | Section 2.6

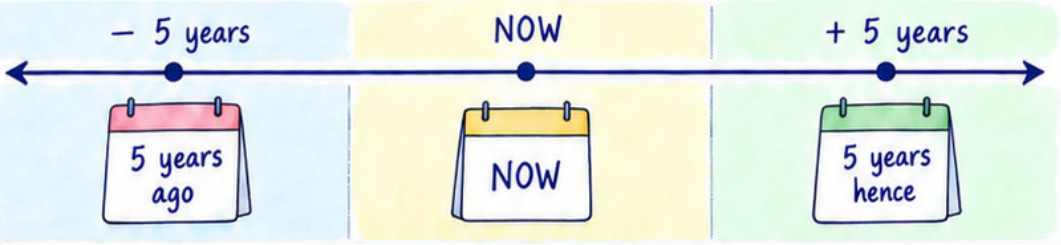
The "n years ago, m years hence" method

When one ratio is given for n years ago and another for m years hence, the two time points are $n+m$ years apart. Write the ages n years ago as ak and bk, add $n+m$ to both, and solve. Then add n to get the present ages.

Worked example

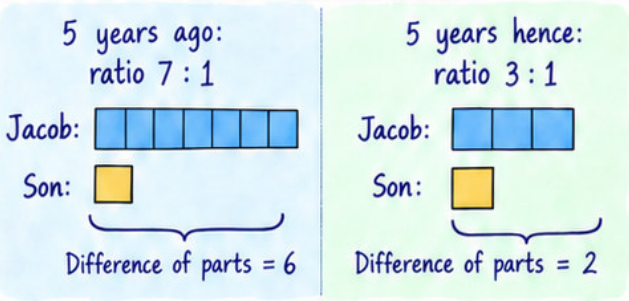
Five years hence, Jacob's age will be three times his son's age; five years ago, it was seven times his son's age. So 5 years ago the ratio was 7 : 1, and 5 years hence it will be 3 : 1; the gap is 10 years. Differences of parts: 6 and 2. Multiply 3 : 1 by 3 to get 9 : 3 (difference 6). The son goes from 1 part to 3 parts: 2 parts = 10 years, so 1 part = 5 years. Five years ago the son was 5 and Jacob 35; now they are 10 and 40. Check: 5 years hence, 45 = 3 × 15. This can also be solved as a pair of linear equations in two unknowns; that method comes in Topic 388 (Day 39).

A clear timeline and ages



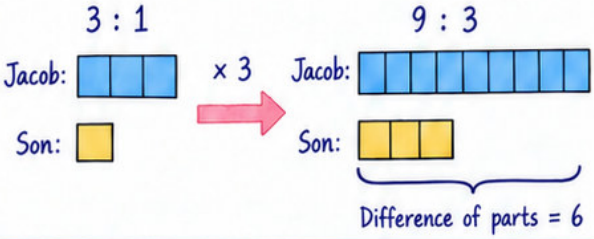
Son's age	5	10	15
Jacob's age	35	40	45

Ratios at the two time points

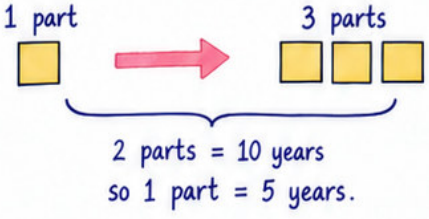


From ratio 3 : 1 to 9 : 3

Multiply 3 : 1 by 3 to get 9 : 3 (difference 6).



The son goes from 1 part to 3 parts



Conclude the ages and check

Five years ago:	Now:	Five years hence:
Son = 5	Son = 10	Son = 15
Jacob = 35	Jacob = 40	Jacob = 45

Check: 5 years hence, 45 = 3 × 15.

Sources: Day 16 Source Index | Topic 158 | Section 2.7

Checking the options by substitution

If the ratio $a : b$ is in its simplest form and the ages are whole numbers, each age at the time of that ratio is a multiple of its part: k = age ÷ part is a whole number.

$a : b$

↓

age = k × part
k = age ÷ part
is a whole number.

$A = ak$

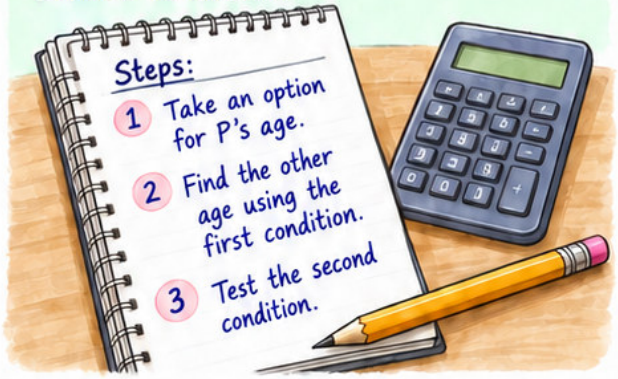
↓

$B = bk$

↓

test the given condition

Then take each remaining option as the age, find the other age from the first condition, and test the second condition.



Example: in the problem of 2.3, let the options for P's age be 21, 24, 28 and 35. 24 is not a multiple of 7, so it goes. For 21, Q is 15, and after 8 years 29 : 23 is not 9 : 7. For 35, Q is 25, and 43 : 33 is not 9 : 7. For 28, Q is 20, and 36 : 28 = 9 : 7. The answer is 28.

P's age (option)	Q's age after first condition	Ratio after 8 years	Verdict
21	15	29 : 23 ≠ 9 : 7	✗ Wrong
24	–	–	✗ Rejected not multiple of 7
28	20	36 : 28 = 9 : 7	✓ Accepted
35	25	43 : 33 ≠ 9 : 7	✗ Wrong

Option 21

Q is 15, and after 8 years 29 : 23 is not 9 : 7.

29 | 23

29 : 23 ≠ 9 : 7 ✗

Option 24

24 is not a multiple of 7, so it goes.

24 ✗

↓

Not a multiple of 7

Option 28

Q is 20, and after 8 years 36 : 28 = 9 : 7.

36 | 28

36 : 28 = 9 : 7 ✓

Option 35

Q is 25, and after 8 years 43 : 33 is not 9 : 7.

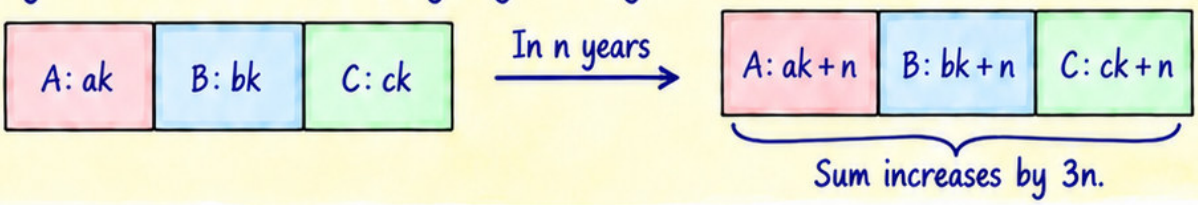
43 | 33

43 : 33 ≠ 9 : 7 ✗

Sources: Day 16 Source Index | Topic 158 | Section 2.8

Three-person problems

Three ages in the ratio $a : b : c$ are ak, bk, and ck.
In n years the sum of three ages grows by $3n$.



If the ratios come in two pieces, join them first by making the common person's number equal (Topic 59).

Example: $A : B = 2 : 3$ and $B : C = 4 : 5$ give $A : B : C = 8 : 12 : 15$.

$A : B = 2 : 3$	$B : C = 4 : 5$	Join the two pieces
Multiply both by 4	Multiply both by 3	$A : B : C = 8 : 12 : 15$
$A : B = 8 : 12$	$B : C = 12 : 15$	

$$A : B : C = 8 : 12 : 15$$

Five years hence the sum of their ages will be 120.
The present sum is $120 - 3 \times 5 = 105$, the parts are $8 + 12 + 15 = 35$, so $k = 3$.
The ages are 24, 36 and 45.
Check: $29 + 41 + 50 = 120$.



Future sum: 120

Present sum: $120 - 3 \times 5 = 105$

Present ages

A	B	C
24	36	45

Ages in 5 years

A	B	C
29	41	50

Check

$29 + 41 + 50 = 120$

Family problems with three or more persons come in Topic 838 (Day 84).

Sources: Day 16 Source Index | Topic 158 | Section 2.9

Which method to use

What the question gives	Method	First step
A ratio now and a ratio n years ago or hence Now: ages in the ratio $ak : bk$ ⟷ (n years) ⟷ n years ago (or hence): ages in a ratio	Ratio parts	Ages ak and bk; add or subtract n for both; cross-multiply.
Two ratios at two time points At one time: a ratio ⟷ (some years) ⟷ At another time: a ratio	Difference of ages	Make the differences of the parts equal; the change in one person's parts equals the years between.
A ratio and the sum of the ages A ratio $ak : bk$ + Sum of ages = S	Sum of ages	Bring the sum to the time of the ratio; divide it by the total of the parts.
A ratio and the difference of the ages A ratio $ak : bk$ + Difference of ages = d	Difference of ages	Difference of parts = the difference in years; no time adjustment.
An average, with a member joining or leaving  joining or leaving	Average of ages	Total = average × number; age the total by (number × years).
One ratio n years ago, another m years hence n years ago: a ratio ⟷ (n + m years) ⟷ m years hence: a ratio	n years ago, m years hence	Unknowns at n years ago; the gap is n + m years.
Four numerical options ☐ 1 ☐ 2 ☐ 3 ☐ 4	Substitution	Keep the multiples of the part; test every condition.

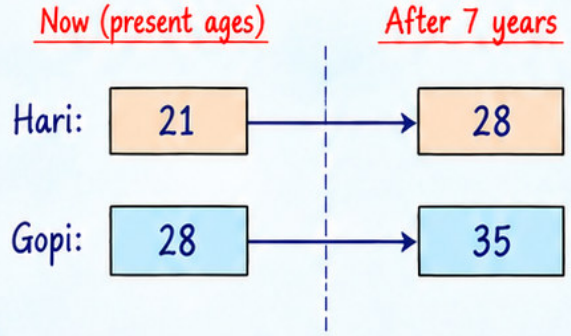
Sources: Day 16 Source Index | Topic 158 | Section 2.10

Extra section: a drill of eight problems with full working (1/3)

1. The present ages of Hari and Gopi are in the ratio 3 : 4. After 7 years the ratio will be 4 : 5. Find their present ages.

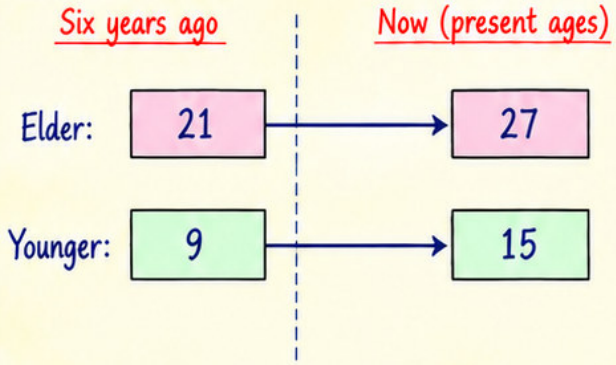
$(3k + 7) : (4k + 7) = 4 : 5$,
so $15k + 35 = 16k + 28$
and $k = 7$.
Hari is 21 and Gopi is 28.
Check: 28 : 35 = 4 : 5.

By differences: both ratios have a difference of 1 part; Hari goes from 3 parts to 4 parts, so 1 part = 7 years.



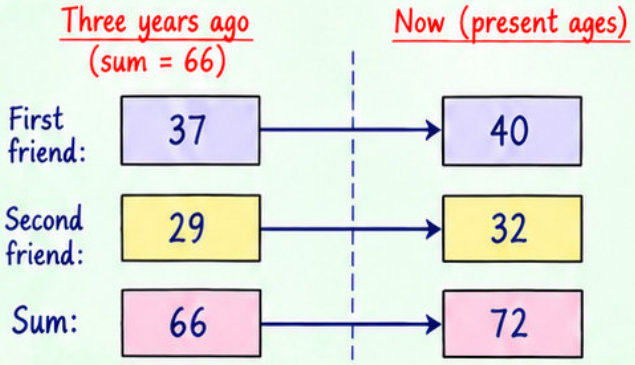
2. The present ages of two colleagues are in the ratio 9 : 5. Six years ago the ratio was 7 : 3. Find their present ages.

Both differences of parts are 4.
The elder goes from 7 parts to 9 parts: 2 parts = 6 years,
so 1 part = 3 years.
Ages now: 27 and 15.
Check: six years ago, 21 : 9 = 7 : 3.



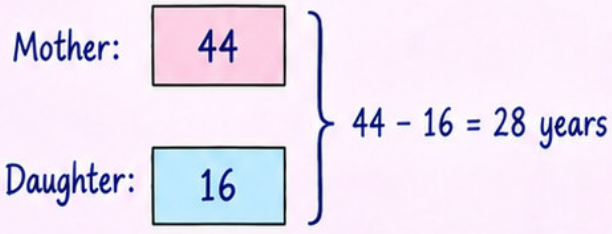
3. The ages of two friends are in the ratio 5 : 4 now. Three years ago the sum of their ages was 66. Find their present ages.

Present sum = $66 + 2 \times 3 = 72$.
9 parts = 72, so $k = 8$.
Ages: 40 and 32.
Check: three years ago, 37 + 29 = 66.



4. The ages of a mother and her daughter are in the ratio 11 : 4, and the mother is 28 years older. Find their ages.

$11 - 4 = 7$ parts = 28 years, so $k = 4$.
Mother 44, daughter 16.
Check: 44 − 16 = 28.



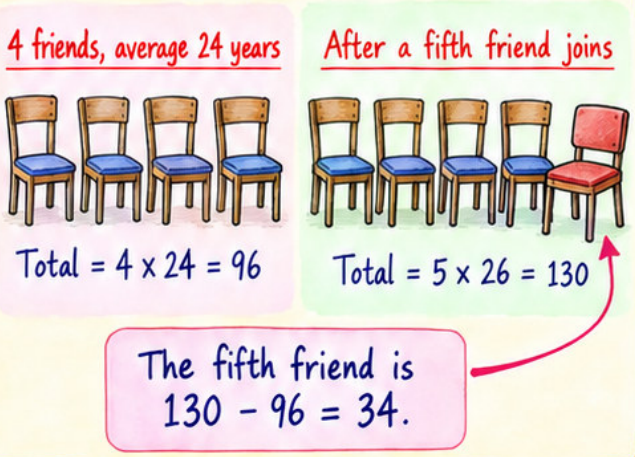
Extra section: a drill of eight problems with full working (2/3)

5. The average age of 4 friends is 24 years. A fifth friend joins them, and the average becomes 26 years. How old is the fifth friend?

Old total = $4 \times 24 = 96$.

New total = $5 \times 26 = 130$.

The fifth friend is $130 - 96 = 34$.



6. Seven years ago the ages of K and L were in the ratio 4 : 3. Seven years hence the ratio will be 6 : 5. Find their present ages.

The gap is $7 + 7 = 14$ years.

Ages 7 years ago: $4k$ and $3k$.

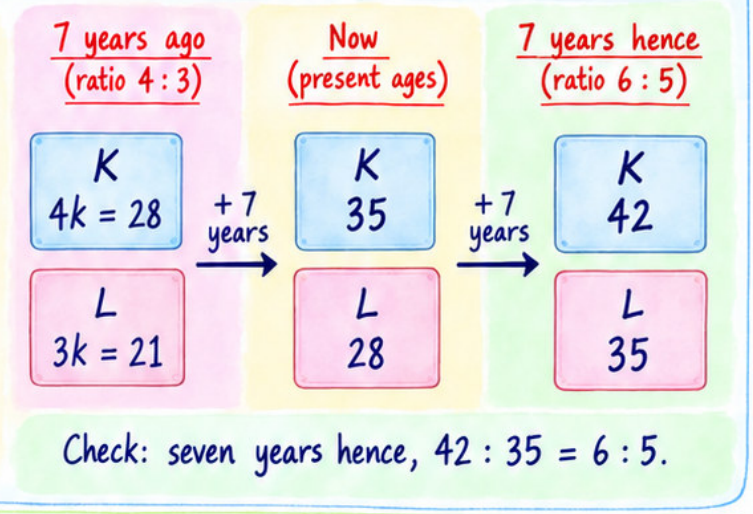
Then $(4k + 14) : (3k + 14) = 6 : 5$,

so $20k + 70 = 18k + 84$ and $k = 7$.

Seven years ago: 28 and 21.

Now: 35 and 28.

Check: seven years hence, $42 : 35 = 6 : 5$.



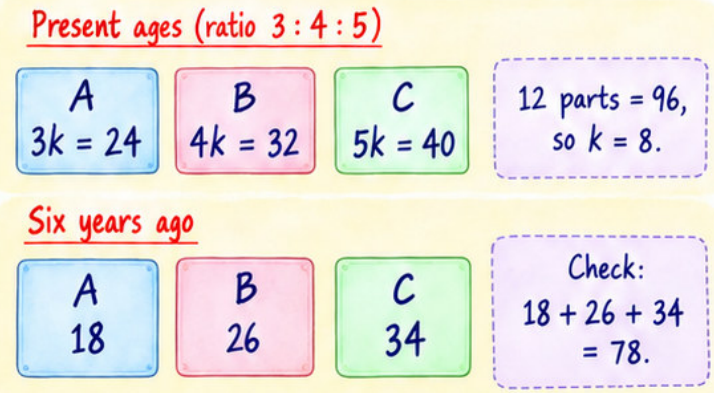
7. The present ages of A, B and C are in the ratio 3 : 4 : 5. Six years ago the sum of their ages was 78. Find their present ages.

Present sum = $78 + 3 \times 6 = 96$.

12 parts = 96, so $k = 8$.

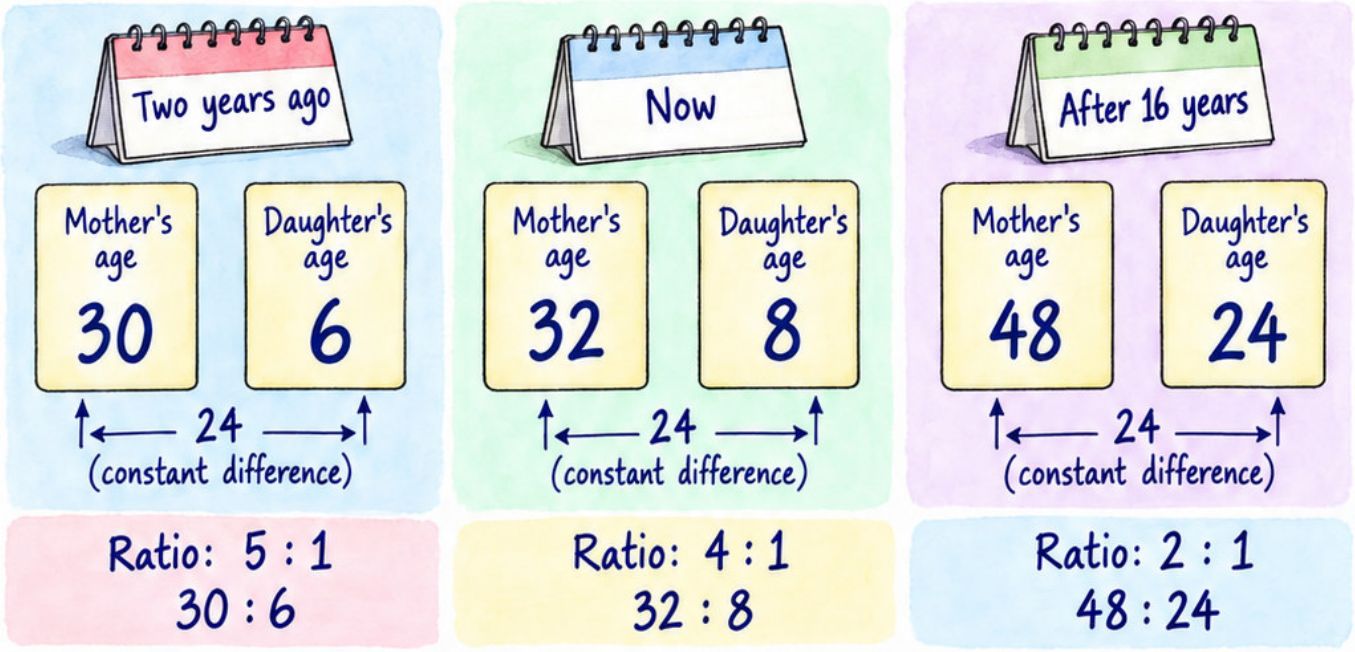
Ages: 24, 32 and 40.

Check: six years ago, $18 + 26 + 34 = 78$.



Extra section: a drill of eight problems with full working (3/3)

8. A mother is 32 and her daughter is 8. After how many years will the mother's age be twice the daughter's?



Ratio: 5 : 1
30 : 6

Ratio: 4 : 1
32 : 8

Ratio: 2 : 1
48 : 24

Solution

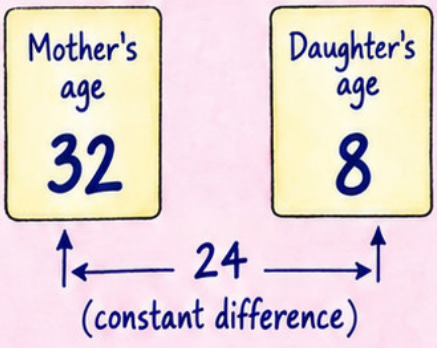
The difference, 24 years, never changes.
At the ratio 2 : 1 the difference is 1 part, so the daughter will then be 24.
She is 8 now, so the answer is
$24 - 8 = 16$ years.

Check: $(32 + 16) : (8 + 16) = 48 : 24 = 2 : 1$.

In the same way, the ratio was 5 : 1 two years ago: 30 : 6.

Key point

The difference, 24 years, never changes.



Sources: Day 16 Source Index | Topic 158 | Section 3

Three common exam traps

1. Every person grows older by the same number of years.

Add (or subtract) the years for each person, not for one of them. The sum of two ages grows by twice the years: two ages that add up to 44 now will add up to 58 after 7 years, not 51. The total of a family grows by (number of members × years).

Two people, sum of ages

Sum of ages now **44** → After 7 years → Sum of ages $44 + 2 \times 7 = 58$

✗ $44 + 7 = 51$ (adds 7 to one person only)

✓ $44 + 2 \times 7 = 58$ (adds 7 to each person)

Total of a family

Number of members n × Years each grows y = Increase in total of ages $n \times y$

Now Total = T → After y years → Later Total = $T + n \times y$

2. Ratio parts are not ages, and a ratio belongs to its own time point.

"Ages in the ratio 8 : 3" means 8k and 3k years, not 8 and 3 years. "Seven years hence the ratio will be 3 : 2, and the difference is 8 years" gives the ages seven years hence, 24 and 16; the present ages are 17 and 9, found by subtracting 7 from each.

Ratio parts are not ages

✗ "Ages in the ratio 8 : 3" means 8 and 3 years.

8 years | 3 years

✓ "Ages in the ratio 8 : 3" means 8k and 3k years, not 8 and 3 years.

8k years | 3k years

Future ratio and difference

Seven years hence: Ratio 3 : 2, Difference = 8 years → Ages in 7 years: 24 and 16 (since 24 − 16 = 8)

↓ Present ages: subtract 7 from each

$24 - 7 = 17$ and $16 - 7 = 9$

3. Answer for the person and the time point that the question asks.

A problem may be solved through one person's age at one time and then ask about the other person, or about another time. If you find that B is 21 now, "B's age 9 years ago" is 12, not 21. Read the last line of the question again before you choose an option.

Find the correct time point

B's age now **21** → 9 years ago → B's age 9 years ago $21 - 9 = 12$

Choose the required answer

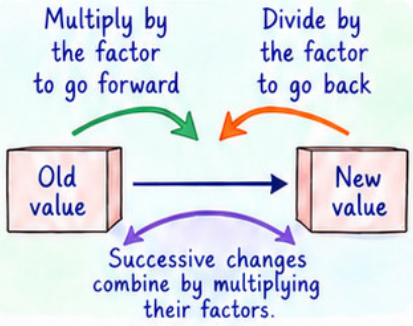
✗ "B's age 9 years ago" is 21. **21**

✓ "B's age 9 years ago" is 12, not 21. **12**

Sources: Day 16 Source Index | Topic 158 | Section 4

Percentage change and percentage points

A percentage change measures a change against the old value, and every rise or fall can be written as a multiplying factor: going forward you multiply by the factor, going back you divide by it, and successive changes combine by multiplying their factors. The exam tests whether you keep the right base and multiply where a hurried candidate adds.



Useful fractions and percentages

Recall from Topic 29: $\frac{1}{2} = 50\%$, $\frac{1}{4} = 25\%$, $\frac{1}{8} = 12\frac{1}{2}\%$, $\frac{1}{3} = 33\frac{1}{3}\%$, $\frac{1}{6} = 16\frac{2}{3}\%$, $\frac{1}{5} = 20\%$ and $\frac{1}{10} = 10\%$. These fractions make most factors in this topic easy to use.

$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{10}$
= 50%	= 25%	$= 12\frac{1}{2}\%$	$= 33\frac{1}{3}\%$	$= 16\frac{2}{3}\%$	= 20%	= 10%

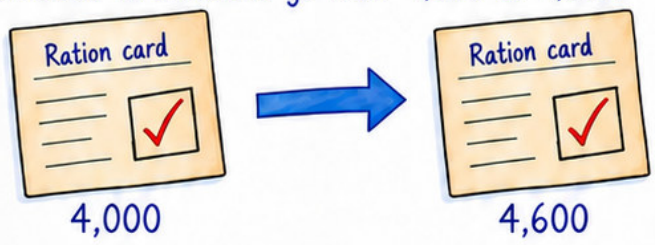
Formula for percentage change

Percentage change = (new value − old value) ÷ old value × 100

(Topic 119). The old value is the base and must be non-zero. For the positive starting quantities in this topic, a positive answer is a rise; a negative answer is a fall.

Example: ration cards in a mandal

Ration cards in a mandal go from 4,000 to 4,600.



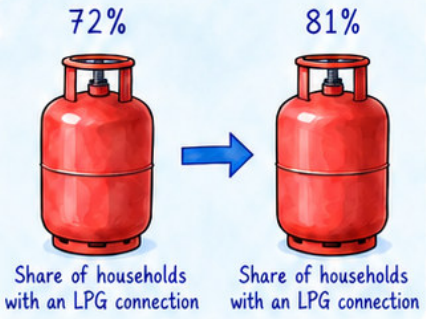
The rise is $\frac{600}{4{,}000} \times 100 = 15\%$.

If they go back from 4,600 to 4,000, the fall is $\frac{600}{4{,}600} \times 100 = 13.04\%$ (approximately).

The gap is the same; the base is not.

Percentage points

When the two figures are themselves percentages, their plain difference is in percentage points (Topics 29 and 119). The share of households with an LPG connection goes from 72% to 81%: a rise of 9 percentage points, which is $\frac{9}{72} \times 100$ = 12.5 per cent of the old share.



Increase
= 81% − 72%
= 9 percentage points

Relative increase
$= \frac{9}{72} \times 100$
= 12.5 per cent of the old share.

Twice a number and 100% fall

Twice a number is a rise of 100%.



This is a rise of 100%.

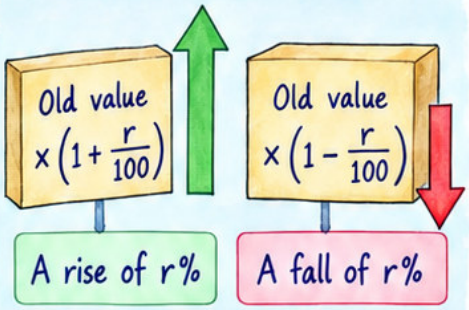
A fall of 100% leaves zero, so a quantity that cannot be negative cannot fall by more than 100%.



Sources: Day 16 Source Index | Topic 159 | Section 2.1

The multiplying factor

A rise of r per cent multiplies a value by $\left(1+\frac{r}{100}\right)$;
a fall of r per cent multiplies it by $\left(1-\frac{r}{100}\right)$.



A rise of 10% on ₹200 gives ₹220, that is, 110% of the old price.



The factor for a rise is the growth factor of Topic 99.

To read a factor back as a change:

change = (factor − 1) × 100 per cent.

A factor of 0.96 is a fall of 4%; a factor of 1.32 is a rise of 32%.

0.96 → −4%
(0.96 − 1) × 100
= −0.04 × 100
= −4%
Fall of 4%

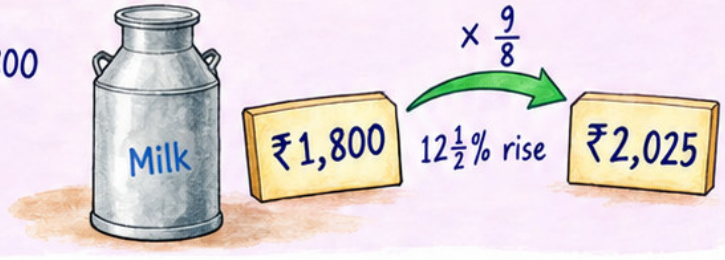
1.32 → +32%
(1.32 − 1) × 100
= 0.32 × 100
= 32%
Rise of 32%

Common percentage changes and their multiplying factors:

Change	Factor as a decimal	Factor as a fraction
Rise of 10%	1.1	11/10
Rise of $12\frac{1}{2}$%	1.125	9/8
Rise of 20%	1.2	6/5
Rise of 25%	1.25	5/4
Rise of $33\frac{1}{3}$%	1.333...	4/3
Rise of 50%	1.5	3/2
Fall of 10%	0.9	9/10
Fall of 20%	0.8	4/5
Fall of 25%	0.75	3/4

Forward change (old to new): new value = old value × factor.

A household's monthly milk bill of ₹1,800 rises by $12\frac{1}{2}$%: $1{,}800 \times \frac{9}{8} = ₹2{,}025$.

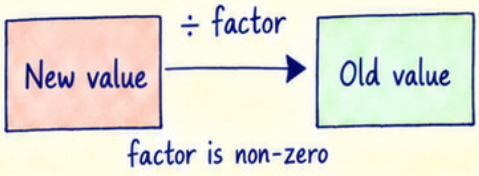


Sources: Day 16 Source Index | Topic 159 | Section 2.2

Reverse change: from the new value back to the old (1/2)

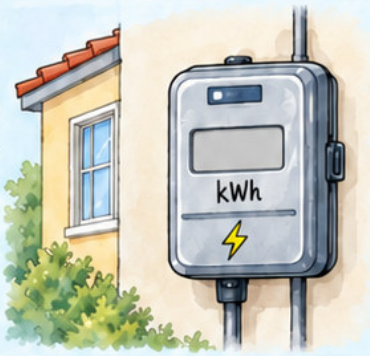
Old value = new value ÷ factor

provided the factor is non-zero. The percentage was a percentage of the old value, so it must not be taken of the new value.



Example:

after a rise of 20%, a family's monthly electricity bill is ₹1,560. The old bill = 1,560 ÷ 1.2 = ₹1,300. Taking 20% off ₹1,560 gives ₹1,248, which is wrong: the 20% was of ₹1,300, not of ₹1,560.



Old bill = 1,560 ÷ 1.2
= ₹1,300 ✓

Taking 20% off ₹1,560 gives ₹1,248, which is wrong. ✗

The 20% was of ₹1,300, not of ₹1,560.

A price after a discount works the same way, because a discount is a fall in price. A pressure cooker sold for ₹2,550 after a discount of 15% had a price of 2,550 ÷ 0.85 = ₹3,000 before the discount. A price that includes a tax of 10% is undone in the same way: ₹3,300 with the tax is 100/110 × 3,300 = ₹3,000 without it. (Discounts in detail: Topic 459.)



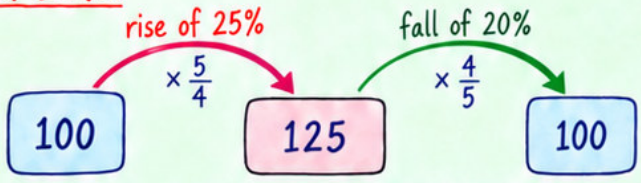
Original price
= 2,550 ÷ 0.85
= ₹3,000



Without tax
$= \frac{100}{110} \times 3{,}300$
= ₹3,000

Why a rise of 25% is undone by a fall of 20%.

The factor of the rise is $\frac{5}{4}$. To undo it, multiply by $\frac{4}{5}$, which is a fall of $\frac{1}{5}$ = 20%. In figures: 100 becomes 125, and 20% of 125 is 25, so 125 comes back to 100.



100 becomes 125.

20% of 125 is 25, so 125 comes back to 100.

In general:

A rise of r% is undone by a fall of $r/(100 + r) \times 100$ per cent, and a fall of r% is undone by a rise of $r/(100 - r) \times 100$ per cent (for $0 \le r < 100$). A 100% fall leaves zero and cannot be reversed by a finite percentage rise.

A rise of r% is undone by a fall of

$$\frac{r}{100 + r} \times 100 \text{ per cent.}$$

A fall of r% is undone by a rise of

$$\frac{r}{100 - r} \times 100 \text{ per cent.}$$

(for $0 \le r < 100$)

A 100% fall leaves zero and cannot be reversed by a finite percentage rise.

Sources: Day 16 Source Index | Topic 159 | Section 2.3

Reverse change: from the new value back to the old (2/2)

A rise undone by a fall

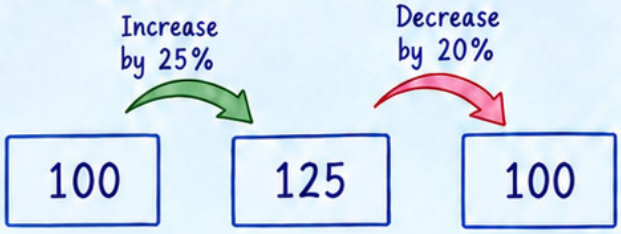
If a quantity increases by r%, it can be brought back to the original value by a fall of the percentage shown.

A fall undone by a rise

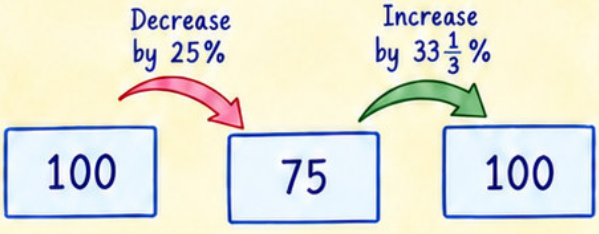
If a quantity decreases by r%, it can be brought back to the original value by a rise of the percentage shown.

r	A rise of r% is undone by a fall of	A fall of r% is undone by a rise of
10	$9\frac{1}{11}\%$	$11\frac{1}{9}\%$
$12\frac{1}{2}$	$11\frac{1}{9}\%$	$14\frac{2}{7}\%$
20	$16\frac{2}{3}\%$	25%
25	20%	$33\frac{1}{3}\%$
$33\frac{1}{3}$	25%	50%
50	$33\frac{1}{3}\%$	100%

Example: a rise of 25%



Example: a fall of 25%



How to use the table

Look up the given percentage r in the first column. The table shows the percentage that undoes the change in each direction.

What this means

Each listed percentage reverses the corresponding original r% change: use the listed fall after an r% rise, or the listed rise after an r% fall.

Note

The percentages are not generally the same in the two directions. A rise of r% is not usually undone by a fall of r%, and a fall of r% is not usually undone by a rise of r%.

Sources: Day 16 Source Index | Topic 159 | Section 2.3

"More than" and "less than"

"A is r% more than B"

"A is r% more than B" takes B as the base. Then B is less than A by $\frac{r}{100+r} \times 100$ per cent, because now A is the base.

This is the rule of 2.3 in other words.

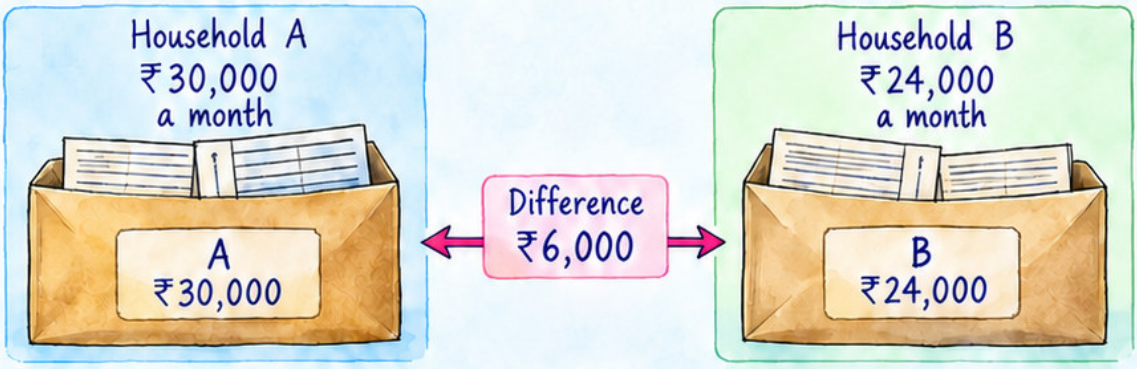
"A is r% less than B"

"A is r% less than B" gives: B is more than A by $\frac{r}{100-r} \times 100$ per cent (for $0 \le r < 100$).

If A is 100% less than B, A is zero and a percentage comparison using A as the base is undefined.

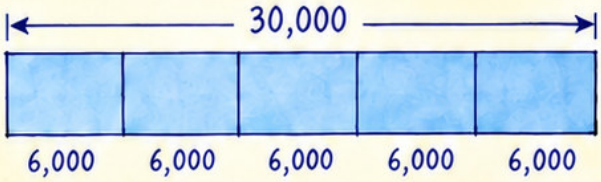
Example:

household A earns ₹30,000 a month and household B ₹24,000.
A earns 6,000/24,000 × 100 = 25% more than B;
B earns 6,000/30,000 × 100 = 20% less than A.
By the rule: 25/125 × 100 = 20.



A is 25% more than B (B → A)

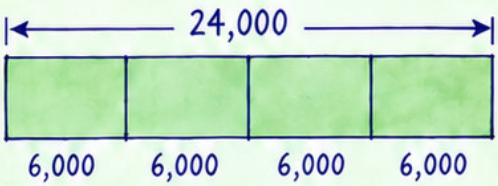
Base = 24,000 (B)



$\frac{6,000}{24,000} \times 100 = 25\%$ more than B.

B is 20% less than A (A → B)

Base = 30,000 (A)



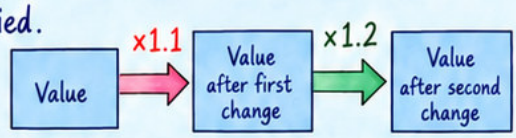
$\frac{6,000}{30,000} \times 100 = 20\%$ less than A.

Check using the rule

By the rule: $\frac{25}{125} \times 100 = 20$.

Successive changes

Each change acts on the value left by the change before it, so the factors are multiplied.



Two changes of a% and b%

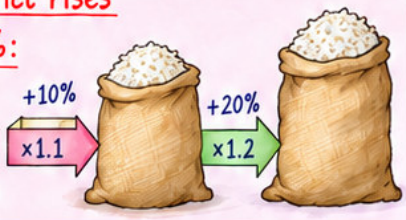
(write a fall with a minus sign):

net change = $a + b + \frac{ab}{100}$ per cent.

Paddy procured in a district rises by 10% and then by 20%:

$10 + 20 + \frac{200}{100} = 32\%$

(check: $1.1 \times 1.2 = 1.32$).

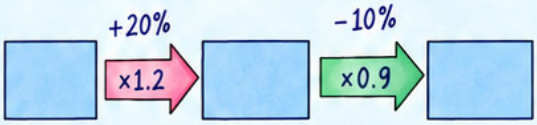


Adding the two gives 30%, which leaves out the second rise acting on the first.

A rise of 20% followed by a fall of 10%:

$20 - 10 - \frac{200}{100} = 8\%$

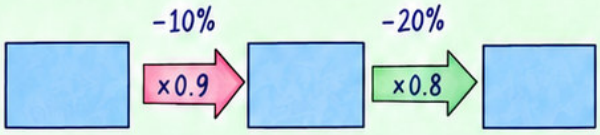
$(1.2 \times 0.9 = 1.08)$.



Two falls of 10% and 20%:

$-10 - 20 + \frac{200}{100} = -28\%$, a fall of 28%

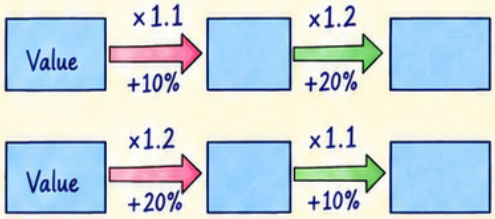
$(0.9 \times 0.8 = 0.72)$.



Written as the size of the fall, it is $a + b - \frac{ab}{100}$.

This minus form is for two falls only; two rises always add the product term.

The order of the changes does not alter the result.



$1.1 \times 1.2 = 1.32$

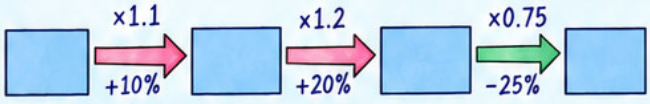
$1.2 \times 1.1 = 1.32$

Three or more changes:

multiply all the factors.

A rise of 10%, a rise of 20% and a fall of 25%:

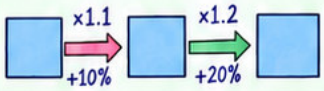
$1.1 \times 1.2 \times 0.75 = 0.99$, a fall of 1%.



Do not replace two different rates by their average.

Rises of 10% and 20% give 1.32, a rise of 32%; two rises of 15% give $1.15 \times 1.15 = 1.3225$, a rise of 32.25%.

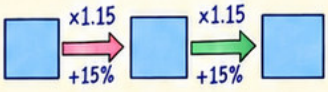
Rises of 10% and 20%



$1.1 \times 1.2 = 1.32$

a rise of 32%

Two rises of 15%



$1.15 \times 1.15 = 1.3225$

a rise of 32.25%

Equal rise and fall.

A rise of r% and a fall of r%, in either order, always give a net fall of $r^2/100$ per cent, because $(1 + \frac{r}{100})(1 - \frac{r}{100}) = 1 - r^2/10{,}000$.

A rise and a fall of 10% leave a fall of 1%; of 20%, a fall of 4%; of 50%, a fall of 25%.

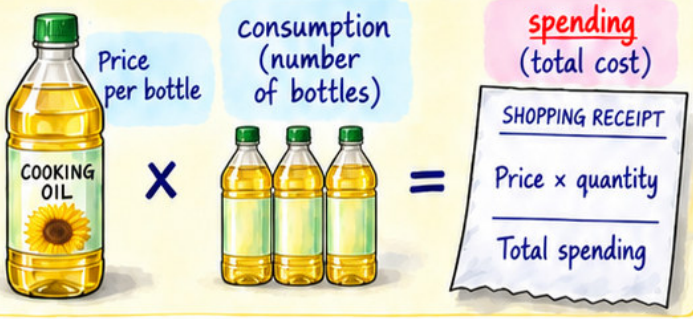
r = 10	r = 20	r = 50
$\frac{10^2}{100} = 1\%$	$\frac{20^2}{100} = 4\%$	$\frac{50^2}{100} = 25\%$
a fall of 1%	a fall of 4%	a fall of 25%

Sources: Day 16 Source Index | Topic 159 | Section 2.5

Price, consumption and spending

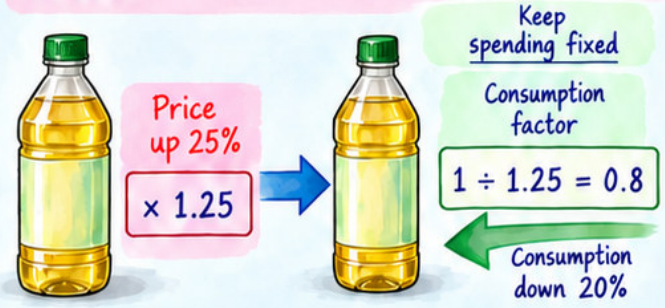
Spending = price × consumption.

So the factor for spending = the factor for price × the factor for consumption.



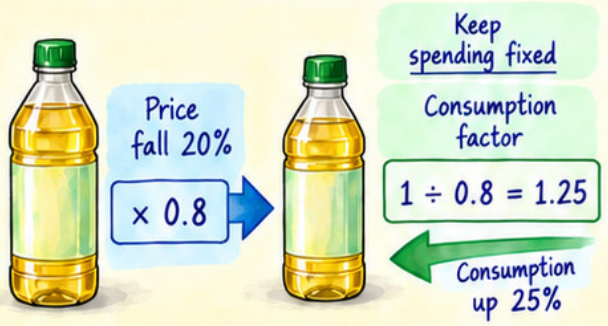
To keep spending fixed after a price rise of r%, the consumption factor must be $1 \div \left(1 + \frac{r}{100}\right)$, so consumption must fall by $\frac{r}{100 + r} \times 100$ per cent.

A rise of 25% in the price of cooking oil needs a cut of $\frac{25}{125} \times 100 = 20\%$ in consumption, not 25%.



After a price fall of r% $(0 \le r < 100)$, consumption can rise by $\frac{r}{100 - r} \times 100$ per cent with spending unchanged.

A fall of 20% in price allows 25% more consumption.



When both change freely, multiply: price up 20% and consumption down 10% give 1.2 × 0.9 = 1.08, so spending rises by 8%.



General formulas

To keep spending fixed after a price rise of r%, the consumption factor must be $1 \div \left(1 + \frac{r}{100}\right)$, so consumption must fall by $\frac{r}{100 + r} \times 100$ per cent.

After a price fall of r% $(0 \le r < 100)$, consumption can rise by $\frac{r}{100 - r} \times 100$ per cent with spending unchanged.

Summary of the examples

Change in price	Change in consumption	Spending factor	Result
up 25%	down 20%	1.25 × 0.8 = 1	spending unchanged
down 20%	up 25%	0.8 × 1.25 = 1	spending unchanged
up 20%	down 10%	1.2 × 0.9 = 1.08	spending rises by 8%

Sources: Day 16 Source Index | Topic 159 | Section 2.6

Compound growth and decline over n periods

The same rate in every period:

value after n periods $= V \times \left(1 + \frac{r}{100}\right)^n$ for growth and $V \times \left(1 - \frac{r}{100}\right)^n$ for decline. This is the formula $A = P\left(1 + \frac{R}{100}\right)^n$ of Topic 99 used for any quantity.

A city of 20,000 people growing at 5% a year



Start 20,000 people → After 1 year × 1.05 → After 2 years × 1.05 → After 3 years × 1.05

has $20{,}000 \times 1.05^3 = 20{,}000 \times 1.157625$
$= 23{,}152.5$, about 23,153 people, after 3 years.

A village of 6,400 people growing at 12½% a year



Start 6,400 people → After 1 year × $\frac{9}{8}$ → After 2 years × $\frac{9}{8}$

has $6{,}400 \times \frac{9}{8} \times \frac{9}{8} = 8{,}100$ people after 2 years.

A two-wheeler bought for ₹80,000 loses 10% of its value every year.



Start ₹80,000 → After 1 year × 0.9 → After 2 years × 0.9

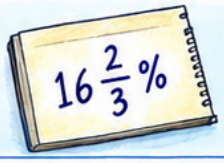
After 2 years it is worth $80{,}000 \times 0.9 \times 0.9 = ₹64{,}800$: a fall of 19%, not 20%.

Different rates in different periods: multiply their factors (2.5).

Start V → First period $\times \left(1 + \frac{r_1}{100}\right)$ → Second period $\times \left(1 + \frac{r_2}{100}\right)$ → Third period $\times \left(1 + \frac{r_3}{100}\right)$ → After 3 periods $V \times \left(1 + \frac{r_1}{100}\right) \times \left(1 + \frac{r_2}{100}\right) \times \left(1 + \frac{r_3}{100}\right)$

Sources: Day 16 Source Index | Topic 159 | Section 2.7

Common traps

Trap	Wrong working	Right working
New value taken as the base 	₹1,560 after a rise of 20%: 1,560 − 312 = ₹1,248 ✗	1,560 ÷ 1.2 = ₹1,300 ✓
Successive changes added 	Rises of 10% and 20%: 30% ✗	1.1 × 1.2 = 1.32: 32% ✓
Minus form used for two rises 	Rises of 10% and 20%: 10 + 20 − 2 = 28% ✗	Two rises add the product term: 32% ✓
Two rates replaced by their average 	Rises of 10% and 20% as 15% twice: 32.25% ✗	32% ✓
Equal rise and fall read as no change 	Up 20%, then down 20%: no change ✗	1.2 × 0.8 = 0.96: a fall of 4% ✓
Consumption cut by the same per cent 	Price up 25%: cut consumption by 25% ✗	Cut by $\frac{25}{125} \times 100 = 20\%$ ✓
A fraction percentage rounded 	A rise of $16\frac{2}{3}\%$ taken as 17% ✗	Use the factor $\frac{7}{6}$ ✓

Other days cover:

Profit and loss (Topic 259)	Price, consumption and expenditure problems (Topic 319)	Marked price and discount (Topic 459)	Compound interest (Topic 529)	Successive discounts (Topic 739)	Population growth and depreciation (Topic 869)
					

Sources: Day 16 Source Index | Topic 159 | Section 2.8

Extra section: a drill of eight questions with full working (1/2)

1. Forward change.

A mandal has 2,400 members in self-help groups, and the number rises by 15%. What is the new number?



10% of 2,400 is 240 and 5% is 120, so the rise is 360.
New number = 2,400 + 360 = 2,760.
With the factor: 2,400 × 1.15 = 2,760.

2. Percentage change.

Households with a toilet in a village go from 3,600 to 4,050. By what per cent did they rise?



Rise = 450.
Base = 3,600, the old value.

$$\frac{450}{3{,}600} = \frac{1}{8} = 12\frac{1}{2}\%.$$

3. Reverse change after a fall.

After a fall of $16\frac{2}{3}\%$ caused by poor rain, a farmer's cotton output is 1,500 kg. What was it before?



$16\frac{2}{3}\% = \frac{1}{6}$, so the factor is $\frac{5}{6}$.

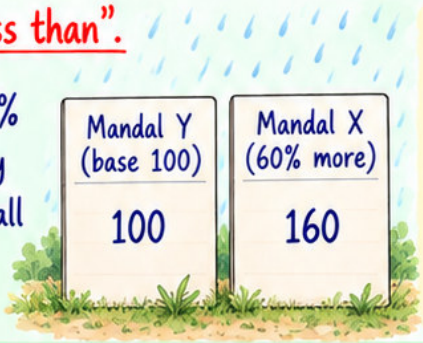
Old output $= 1{,}500 \div \left(\frac{5}{6}\right) = 1{,}500 \times \frac{6}{5}$

$= 1{,}800$ kg.

Adding $16\frac{2}{3}\%$ of 1,500 (that is, 250) gives 1,750 kg, which uses the wrong base.

4. "More than" and "less than".

Rainfall in Mandal X is 60% more than in Mandal Y. By what per cent is the rainfall in Y less than in X?



Take Y as 100; then X is 160.

Y is less by $\frac{60}{160} \times 100 = 37\frac{1}{2}\%$.

By the rule: $\frac{60}{100 + 60} \times 100 = 37\frac{1}{2}\%$.

5. A rise and then a fall.

A household's monthly grocery bill rises by 30% and then falls by 20%. What is the net change?



$30 - 20 - \frac{600}{100} = 4\%$, a rise of 4%.

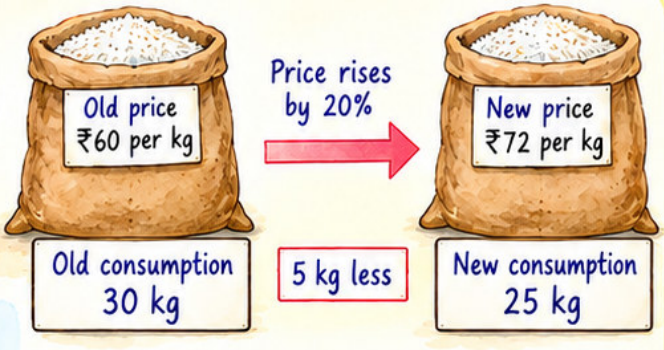
Check: 1.3 × 0.8 = 1.04.

Sources: Day 16 Source Index | Topic 159 | Section 3

Extra section: a drill of eight questions with full working (2/2)

6. Price and consumption.

The price of rice rises by 20%. A household that still spends ₹1,800 a month on rice now gets 5 kg less. What was the old price per kg?



Consumption must fall by $\frac{20}{120} = \frac{1}{6}$.

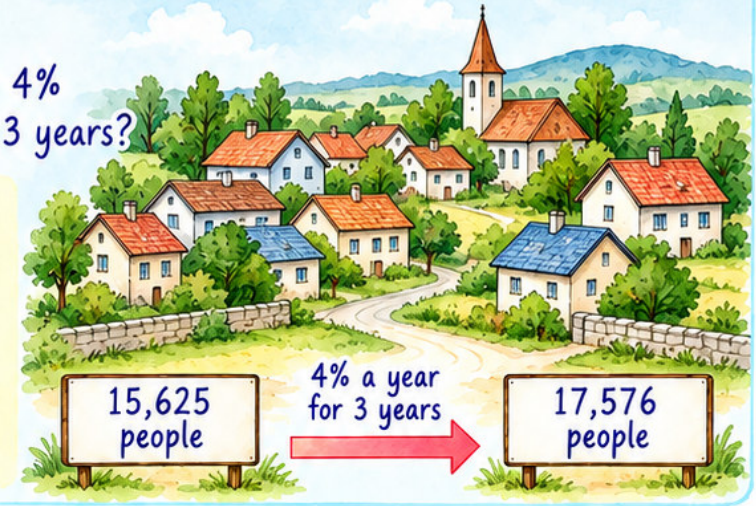
So $\frac{1}{6}$ of the old quantity is 5 kg, and the old quantity is 30 kg.

Old price = 1,800 ÷ 30 = ₹60 per kg.

Check:
the new price is ₹72, and 1,800 ÷ 72 = 25 kg, which is 5 kg less.

7. Compound growth.

A village of 15,625 people grows at 4% a year. What is its population after 3 years?



Factor = 1.04 = $\frac{26}{25}$.

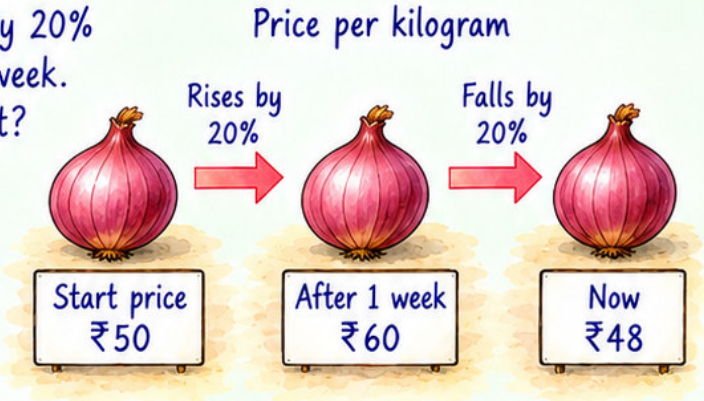
Population = $15{,}625 \times \frac{26}{25} \times \frac{26}{25} \times \frac{26}{25}$

= 17,576

(15,625 is 25 × 25 × 25).

8. Equal rise and fall, then reverse.

The price of a kilogram of onions rises by 20% in one week and falls by 20% the next week. It is now ₹48. What was it at the start?



Net factor = 1.2 × 0.8 = 0.96,

a fall of $\frac{20^2}{100}$ = 4%.

Start price = 48 ÷ 0.96 = ₹50.

Sources: Day 16 Source Index | Topic 159 | Section 3

Three common exam traps

1. In a reverse change, divide by the factor.

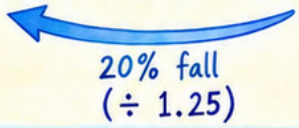
The given percentage is a percentage of the old value.
A figure that stands at 600 after a rise of 20% was $600 \div 1.2 = 500$, not $600 - 120 = 480$.
For the same reason a rise of 25% is undone by a fall of 20%, not of 25%.



So the old value was
$600 \div 1.2 = 500$
$600 - 120 = 480$ ✗
(incorrect)

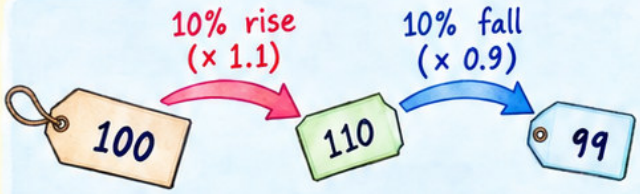


To undo a rise of 25%, we divide by 1.25, which is the same as a fall of 20%.

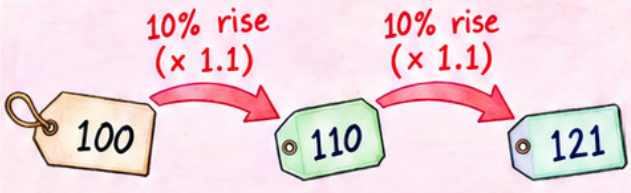


2. Successive changes are multiplied, not added.

A rise of 10% followed by a fall of 10% leaves $1.1 \times 0.9 = 0.99$ of the value, a fall of 1%, not "no change".
Two rises of 10% give 21%, not 20%.



Overall factor = $1.1 \times 0.9 = 0.99$
This is a fall of 1%, not "no change".



Overall factor = $1.1 \times 1.1 = 1.21$
This is a rise of 21%, not 20%.

3. A price rise is offset by a smaller cut in consumption.

To keep spending fixed after a price rise of 50%, consumption must fall by $50/150 \times 100 = 33\frac{1}{3}\%$, not by 50%.



Initial situation
Price = 100
Consumption = 100
Spending = 100 × 100
= 10 000

Price rises by 50%
(× 1.5)



To keep spending fixed
New price = 150
New consumption = x
Spending = $150 \times x$
Set $150 \times x = 10\,000$
so $x = 10\,000 \div 150$
$= 66\frac{2}{3}$

Percentage fall in consumption
$\frac{(100 - 66\frac{2}{3})}{100} \times 100$
$= \frac{50}{150} \times 100$
$= 33\frac{1}{3}\%$,
not by 50%.